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″ *Never shoot, Never hit* ″

THE LAST APPROACH TO THE SETTLEMENT OF THE JACOBIAN CONJECTURE

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Abstract. The Jacobian Conjecture can be generalized and is established : Let S be a polynomial ring over a field of characteristic zero in finitely may variables. Let T be an unramified, finitely generated extension of S with $T^\times = k^\times$. Then $T = S$.

Let k be an algebraically closed field, let $\mathbb{A}_k^n$ be an affine space of dimension n over k and let $f : \mathbb{A}_k^n \longrightarrow \mathbb{A}_k^n$ be a morphism of affine spaces over k of dimension n. Then f is given by coordinate functions $f_1, \ldots, f_n$, where $f_i \in k[X_1, \ldots, X_n]$ and $\mathbb{A}_k^n = Max(k[X_1, \ldots, X_n])$. If f has an inverse morphism, then the Jacobian $\det(\partial f_i/\partial X_j)$ is a nonzero constant. This follows from the easy chain rule. The Jacobian Conjecture asserts the converse.

If k is of characteristic $p > 0$ and $f(X) = X + X^p$, then $df/dX = f'(X) = 1$ but X can not be expressed as a polynomial in f. Thus we must assume the characteristic of k is zero.

The Jacobian Conjecture of geometric form. *Let $f : \mathbb{A}_k^n \to \mathbb{A}_k^n$ be a morphism of affine spaces of dimension n $(n \geq 1)$ over a field of characteristic zero. Then f is expressed by coordinate functions $f_1, \ldots, f_n$, where $f_i \in k[X_1, \ldots, X_n]$. If the Jacobian $\det(\partial f_i/\partial X_j)$ is a nonzero constant, then f is an isomorphism.*

The algebraic form of the Jacobian Conjecture is the following :

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The Jacobian Conjecture of algebraic form. *If $f_1, \cdots, f_n$ be elements in a polynomial ring $k[X_1, \cdots, X_n]$ over a field k of characteristic zero such that $\det(\partial f_i/\partial X_j)$ is a nonzero constant, then $k[f_1, \cdots, f_n] = k[X_1, \cdots, X_n]$.*

To prove the Jacobian Conjecture, we treat a more general case. More precisely, we show the following result:

Let k be a algebraically closed field of characteristic zero, let S be a polynomial ring over k of finite variables and let T be an unramified, finitely generated extension domain of S with $T^\times = k^\times$. Then $T = S$.

Throughout this paper, all fields, rings and algebras are assumed to be commutative with unity. For a ring R, $R^\times$ denotes the set of units of R and $K(R)$ the total quotient ring. $\mathrm{Spec}(R)$ denotes the affine scheme defined by R or merely the set of all prime ideals of R and $\mathrm{Ht}_1(R)$ denotes the set of all prime ideals of height one. Our general reference for unexplained technical terms is [**9**].

1. Preliminaries

Definition. Let $f : A \to B$ be a ring-homomorphism of finite type of locally Noetherian rings. The homomorphism f is called *unramified* if $PB_P = (P \cap A)B_P$ and $k(P) = B_P/PB_P$ is a finite separable field extension of $k(P \cap A) = A_{P\cap A}/(P \cap A)A_{P\cap A}$ for all prime ideal P of B. The homomorphism f is called *etale* if f is unramified and flat.

Proposition 1.1. *Let k be an algebraically closed field of characteristic zero and let B be a polynomial ring $k[Y_1, \ldots, Y_n]$. Let L be a finite Galois extension of the*

quotient field of B and let D be an integral closure of B in L. If D is etale over B then $D = B$.

Proof. We may assume that $k = \mathbf{C}$, the field of complex numbers by "Lefschetz Principle" (cf.[**4, p.290**]). The extension D/B is etale and finite, and so

$$\mathrm{Max}(D) \to \mathrm{Max}(B) \cong \mathbf{C}^n$$

is a (connected) covering. Since $\mathbf{C}^n$ is simply connected, we have $D = B$. (An algebraic proof of the simple connectivity of k^n is seen in [**15**].) □

Recall the following well-known results, which are required for proving Theorem 2.1 below.

Lemma A ([**9,(21.D)**]). *Let (A, m, k) and (B, n, k') be Noetherian local rings and $\phi : A \to B$ a local homomorphism (i.e., $\phi(m) \subseteq n$). If $\dim B = \dim A + \dim B \otimes_A k$ holds and if A and $B \otimes_A k = B/mB$ are regular, then B is flat over A and regular.*

Proof. If $\{ x_1, \ldots, x_r \}$ is a regular system of parameters of A and if $y_1, \ldots, y_s \in n$ are such that their images form a regular system of parameters of B/mB, then $\{ \varphi(x_1), \ldots, \varphi(x_r), y_1, \ldots, y_s \}$ generates n. and $r + s = \dim B$. Hence B is regular. To show flatness, we have only to prove $\mathrm{Tor}_1^A(k, B) = 0$. The Koszul complex $K_*(x_1, \ldots, x_r; A)$ is a free resolution of the A-module k. So we have $\mathrm{Tor}_1^A(k, B) = H_1(K_*(x_1, \ldots, x_r; A) \otimes_A B) = H_1(K_*(x_1, \ldots, x_r; B))$. Since the sequence $\varphi(x_1), \ldots, \varphi(x_r)$ is a part of a regular system of parameters of B, it is a B-regular sequence. Thus $H_i(K_*(x_1, \ldots, x_r; B)) = 0$ for all $i > 0$. □

Corollary A.1. *Let k be a field and let $R = k[X_1, \ldots, X_n]$ be a polynomial ring. Let S be a finitely generated ring-extension of R. If S is unramified over R, then S is etale over R.*

Proof. We have only to show that S is flat over R. Take $P \in \mathrm{Spec}(S)$ and put $p = P \cap R$. Then $R_p \hookrightarrow S_P$ is a local homomorphism. Since S_P is unramified over

R_p, we have $\dim S_P = \dim R_p$ and $S_P \otimes_{R_p} k(p) = S_P/PS_P = k(P)$ is a field. So by Lemma A, S_P is flat over R_p. Therefore S is flat over R by **[5,p.91]**. $\square$

Example. Let k be a field of characteristic $p > 0$ and let $S = k[X]$ be a polynomial ring. Let $f = X + X^p \in S$. Then the Jacobian matrix $\left(\dfrac{\partial f}{\partial X}\right)$ is invertible. So $k[f] \hookrightarrow k[X]$ is finite and unramified. Thus $k[f] \hookrightarrow k[X]$ is etale by Corollary A.1. Indeed, it is easy to see that $k[X] = k[f] \oplus Xk[f] \oplus \cdots \oplus X^{p-1}k[f]$ as a $k[f]$-module, which implies that $k[X]$ is free over $k[f]$.

Lemma B (**[2,Chap.V, Theorem 5.1]**). *Let A be a Noetherian ring and B an A-algebra of finite type. If B is flat over A, then the canonical map* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is an open map.*

Lemma C (**[10, p.51,Theorem 3']**). *Let k be a field and let V be a k-affine variety defined by a k-affine ring R (which means a finitely generated algebra over k) and let F be a closed subset of V defined by an ideal I of R. If the variety $V \setminus F$ is k-affine, then F is pure of codimension one.*

Lemma D(**[16,Theorem 9, § 4, Chap.V]**). *Let k be a field, let R be a k-affine domain and let L be a finite algebraic field extension of $K(R)$. Let R_L denote the integral closure of R in L. Then R_L is a module finite type over R.*

Lemma E(**[12, Ch.IV,Corollary 2]**)(Zariski's Main Theorem). *Let A be an integral domain and let B be an A-algebra of finite type which is quasi-finite over A. Let $\overline{A}$ be the integral closure of A in B. Then the canonical morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(\overline{A})$ *is an open immersion.*

Lemma F(**[3, Corollary 7.10]**). *Let k be a field, A a finitely generated k-algebra. Let M be a maximal ideal of A. Then the field A/M is a finite algebraic extension of k. In particular, if k is algebraically closed then $A/M \cong k$.*

Lemma G (**[2,VI(3.5)]**). *Let $f : A \to B$ and $g : B \to C$ be ring-homomorphisms of finite type of locally Noetherian rings.*

(i) *Any immersion ${}^a f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is unramified.*

(ii) *The composition* $g \cdot f$ *of unramified homomorphisms* f *and* g *is unramified.*

(iii) *If* $g \cdot f$ *is an unramified homomorphism, then* g *is an unramified homomorphism.*

Lemma H ([**2,VI(4.7)**]). *Let* $f : A \to B$ *and* $g : B \to C$ *be ring-homomorphisms of finite type of locally Noetherian rings.* B *(resp.* C*) is considered to be an* A*-algebra by* f *(resp.* $g \cdot f$*).*

(i) *The composition* $g \cdot f$ *of etale homomorphisms* f *and* g *is etale.*

(ii) *Any base-extension* $f \otimes_A 1_C : C = A \otimes_A C \to B \otimes_A C$ *of an etale homomorphism* f *is etale.*

(iii) *If* $g \cdot f : A \to B \to C$ *is an etale homomorphism and if* f *is an unramified homomorphism, then* g *is etale.*

Corollary H.1. *Let* R *be a ring and let* $B \to C$ *and* $D \to E$ *be etale* R*-algebra homomorphisms. Then the homomorphism* $B \otimes_R D \to C \otimes_R E$ *is an etale homomorphism.*

Proof. The homomorphism

$$B \otimes_R D \to B \otimes_R E \to C \otimes_R E$$

is given by composite of base-extensions. So by Lemma H, this composite homomorphism is etale. $\square$

Lemma I ([**11,(41.1)**])(Purity of branch loci). *Let* R *be a regular ring and let* A *be a normal ring which is a finite extension of* R*. Assume that* $K(A)$ *is finite separable extension of* $K(R)$*. If* A_P *is unramified over* $R_{P \cap R}$ *for all* $P \in \mathrm{Ht}_1(A)(= \{Q \in \mathrm{Spec}(A) | \mathrm{ht}(Q) = 1\})$*, then* A *is unramified over* R*.*

Lemma J (cf. [**17,(1.3.10)**]). *Let* S *be a scheme and let* (X, f) *and* (Y, g) *be* S*-schemes. For a scheme* Z*,* $|Z|$ *denotes its underlying topological space. Let* $p : X \times_S Y \to X$ *and* $q : X \times_S Y \to Y$ *be projections. Then the map of topological spaces* $|p| \times_{|S|} |q| : |X \times_S Y| \to |X| \times_{|S|} |Y|$ *is a surjective map.*

Proof. Let $x \in X$, $y \in Y$ be points such that $f(x) = g(y) = s \in S$. Then the residue class fields $k(x)$ and $k(y)$ are the extension-fields of $k(s)$. Let K denote an extension-field of $k(s)$ containing two fields which are isomorphic to $k(x)$ and $k(y)$. Such field K is certainly exists. For instance, we have only to consider the field $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s)}} \mathcal{O}_{Y,y}/m$, where m is a maximal ideal of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s)}} \mathcal{O}_{Y,y}$. Let $x_K : \mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \xrightarrow{i_x} X$, where i_x is the canonical immersion as topological spaces and the identity $i_x^*(\mathcal{O}_X) = \mathcal{O}_{X,x}$ as structure sheaves. Let y_K be the one similarly defined as x_K. By the construction of x_K, y_K, we have $f \cdot x_K = g \cdot y_K$. Thus there exists a S-morphism $z_K : \mathrm{Spec}(K) \to X \times_S Y$ such that $p \cdot z_K = x_K$, $q \cdot z_K = y_K$. Since $\mathrm{Spec}(K)$ consists of a single point, putting its image $= z$, we have $p(z) = x$, $q(z) = y$. Therefore the map of topological spaces $|p| \times_{|S|} |q| : |X \times_S Y| \to |X| \times_{|S|} |Y|$ is surjective.

$\square$

Remark 1.1. Let $A \to B$ be a ring-homomorphism of rings. Let $pr_i : \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \to \mathrm{Spec}(B)$ $(i = 1, 2)$ be the projection. Recall that an affine scheme $\mathrm{Spec}(B)$ is separated over $\mathrm{Spec}(A)$, that is, the diagonal morphism $\Delta : \mathrm{Spec}(B) \to \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ (defined by $B \otimes_A B \ni x \otimes y \mapsto xy \in B$) is a closed immersion and $pr_i \cdot \Delta = id_{\mathrm{Spec}(B)}$ $(i = 1.2)$ (cf. **[17]**). It is easy to see that the diagonal morphism $\Delta' : \mathrm{Spec}(B) \to \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \cdots \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ (n-times) similarly defined is also a closed immersion with $p_i \cdot \Delta' = id_{\mathrm{Spec}(B)}$, where pr_i is the projection $(1 \le i \le n)$. Let $B_2, \ldots, B_n$ be A-algebras such that $B \cong_A B_2 \cong_A \cdots \cong_A B_n$. Then there exists a $\mathrm{Spec}(A)$-morphism $\Delta^* : \mathrm{Spec}(B) \to \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \cdots \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \cong_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B_2) \times_{\mathrm{Spec}(A)} \cdots \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B_n)$, which is a closed immersion and $pr_1 \cdot \Delta^* = id_{\mathrm{Spec}(B)}$. Hence pr_1 is surjective.

Remark 1.2. Let k be a field, let $S = k[Y_1, \ldots, Y_n]$ be a polynomial ring over k and let L be a finite Galois extension field of $K(S)$ with Galois group $G = \{\ \sigma_1 = 1, \sigma_2, \ldots, \sigma_\ell\}$. Let T be a finitely generated, flat extension of S contained in L

with $T^\times = k^\times$. Put $T^{\sigma_i} = \sigma_i(T) \subseteq L$. Let

$$T^\# := T^{\sigma_1} \otimes_S \cdots \otimes_S T^{\sigma_\ell},$$

which has the natural T-algebra structure by $T \otimes_S S \otimes_S \cdots \otimes_S S \hookrightarrow T^{\sigma_1} \otimes_S \cdots \otimes_S T^{\sigma_\ell} = T^\#$.

(i) Let P be a prime ideal of T. Then the element $(P^{\sigma_1}, \ldots, P^{\sigma_\ell}) \in |\mathrm{Spec}(T^{\sigma_1})| \times_{|\mathrm{Spec}(S)|} \cdots \times_{|\mathrm{Spec}(S)|} |\mathrm{Spec}(T^{\sigma_\ell})|$ is an image of some element Q in $|\mathrm{Spec}(T^\#)|$ because the canonical map $|\mathrm{Spec}(T^\#)| = |\mathrm{Spec}(T^{\sigma_1} \otimes_S \cdots \otimes_S T^{\sigma_\ell})| \to |\mathrm{Spec}(T^{\sigma_1})| \times_{|\mathrm{Spec}(S)|} \cdots \times_{|\mathrm{Spec}(S)|} |\mathrm{Spec}(T^{\sigma_\ell})|$ is surjective by Lemma J. The map $|\mathrm{Spec}(T^\#)| \to |\mathrm{Spec}(T)|$ yields that $Q \cap T = P$ Hence $|\mathrm{Spec}(T^\#)| \to |\mathrm{Spec}(T)|$ is surjective. (This result has been obtained in Remark 1.1.) So $T^\#$ is faithfully flat over T.

(ii) Take $p \in \mathrm{Ht}_1(S)$. Then p is a principal ideal of S and so $pT^{\sigma_i} \neq T^{\sigma_i}$ ($\forall \sigma_i \in G$) because $T^\times = k^\times$. Let P be a minimal prime divisor of pT. Then $P^{\sigma_i} \in \mathrm{Spec}(T^{\sigma_i})$ and $P^{\sigma_i} \cap S = p$ because $S \hookrightarrow T$ is flat. There exists a prime ideal Q in $\mathrm{Spec}(T^\#)$ with $Q \cap T = P$ by (i) and hence $P \cap S = p$. Thus $Q \cap S = p$. Therefore $pT^\# \neq T^\#$ for all $p \in \mathrm{Ht}_1(S)$.

2. Main Result

The following is our main theorem.

Theorem 2.1. *Let k be a algebraically closed field of characteristic zero, let S be a polynomial ring over k of finitely many variables and let T be an unramified, finitely generated extension domain of S with $T^\times = k^\times$. Then $T = S$.*

Proof.

(1) Let $K(\ \ )$ denote the quotient field of $(\ \ )$. There exists a minimal finite Galois extension L of $K(S)$ containing T because $K(T)/K(S)$ is a finite algebraic extension.

Let G be the Galois group $G(L/K(S))$. Put $G = \{\ \sigma_1 = 1, \sigma_2, \ldots, \sigma_\ell\}$, where $\sigma_i \neq \sigma_j$ if $i \neq j$. Put $T^\sigma := \sigma(T)$ ($\forall\ \sigma \in G$) and put $D := S[\bigcup_{\sigma\in G} T^\sigma] = S[\bigcup_{i=1}^{\ell} T^{\sigma_i}] \subseteq L$. Then $K(D) = L$ since L is a minimal Galois extension of $K(S)$ containing $K(T)$. Since $\mathrm{Spec}(T) \to \mathrm{Spec}(S)$ is etale (Corollary A.1 or [**4, p.296**]), so is $\mathrm{Spec}(T^\sigma) \to \mathrm{Spec}(S)$ for each $\sigma \in G$.

Put

$$T^{\#} := T^{\sigma_1} \otimes_S \cdots \otimes_S T^{\sigma_\ell},$$

which has the natural T-algebra structure by $T = T \otimes_S S \otimes_S \cdots \otimes_S S \hookrightarrow T^{\sigma_1} \otimes_S \cdots \otimes_S T^{\sigma_\ell} = T^{\#}$. This homomorphism is etale by Corollary H.1 because $S \to T$ is etale. Let $\psi' : T^{\#} = T^{\sigma_1} \otimes_S \cdots \otimes_S T^{\sigma_\ell} \to L$ be an S-algebra homomorphism sending $a_1^{\sigma_1} \otimes \cdots \otimes a_\ell^{\sigma_\ell}$ to $a_1^{\sigma_1} \cdots a_\ell^{\sigma_\ell}$ ($a_i \in T$). Then $D = \mathrm{Im}(\psi') = S[\bigcup_{\sigma\in G} T^\sigma] \subseteq L$. Since $\mathrm{Spec}(T) \to \mathrm{Spec}(S)$ is etale, the canonical morphism $\mathrm{Spec}(T^{\#}) = \mathrm{Spec}(T^{\sigma_1} \otimes_S \cdots \otimes_S T^{\sigma_\ell}) \to \mathrm{Spec}(T^{\sigma_1} \otimes_S S \otimes_S \cdots \otimes_S S) = \mathrm{Spec}(T)$ is etale, and the natural surjection $\psi : T^{\#} = T^{\sigma_1} \otimes_S \cdots \otimes_S T^{\sigma_\ell} \to D$ is unramified by Lemma G(i)(or [**2,VI(3.5)**]). So $[T \hookrightarrow D] = [T \hookrightarrow T^{\#} \to D]$ is unramified by Lemma G(ii) because etale is flat and unramified. Moreover $S \hookrightarrow T \hookrightarrow D$ is also unramified. Since T and D are unramified over S, both T and D are etale over S and both T and D are regular by Corollary A.1.

Let $I := \mathrm{Ker}\psi$. So ${}^a\psi : \mathrm{Spec}(D) \cong V(I) \subseteq \mathrm{Spec}(T^{\#})$ is a closed immersion. Since $[T \hookrightarrow T^{\#} \to D] = [T \hookrightarrow D]$ is etale, so is $\psi : T^{\#} \to D$ by Lemma H(iii) (or [**2,VI(4.7)**]). It follows that $\mathrm{Spec}(D) \to \mathrm{Spec}(T^{\#})$ is a closed immersion and an open map because a flat morphism is an open map by Lemma B. Thus $\mathrm{Spec}(D) = V(I) \subseteq \mathrm{Spec}(T^{\#})$ is a connected component of $\mathrm{Spec}(T^{\#})$. So we have seen that the natural S-homomorphism $T \hookrightarrow T^{\#} \to D$ is etale and that $\mathrm{Spec}(D)$ is a connected component of $\mathrm{Spec}(T^{\#})$. Note that $T^{\#}$ is reduced because $T^{\#}$ is unramified over S, and that $\dim S = \dim T = \dim D$ because S, T and D are all k-affine domains with the same transcendence degree over k.

Let $(0) = \bigcap_{i=1}^{s} P_i$ be an irredundant primary decomposition in $T^{\#}$. Since $T \to T^{\#}$ is flat, the GD-theorem [**9,(5.D)**](or Lemma B) holds for this homomorphism $T \to T^{\#}$. In the decomposition $(0) = \bigcap_{i=1}^{s} P_i$, each P_i is a minimal prime divisor of (0), so we have $T \cap P_i = (0)$ for all $i = 1, \ldots, s$. Note that $S \hookrightarrow T^{\sigma_i}$ is unramified

and hence that $T^\#$ is reduced. The P_i's are prime ideals of $T^\#$. Note that I is a prime ideal of $T^\#$ and that $\dim S = \dim T = \dim T^\sigma = \dim D$ for each $\sigma \in G$. Thus there exists j, say $j = 1$, such that $I = P_1$. In this case, $P_1 + \bigcap_{i=2}^s P_i = T^\#$ and $T^\#/P_1 \cong D \subseteq L$ as T-algebra. Note that T is considered to be a subring of $T^\#$ by the canonical injective homomorphisms $T = T \otimes_S S \otimes_S \cdots \otimes_S S \hookrightarrow T^\#$ and that $[T \hookrightarrow T^\# \to T^\#/P_1 \cong D] = [T \hookrightarrow D]$. Putting $C = T^\#/\bigcap_{i=2}^s P_i$, we have $T^\# \overset{\Phi}{\tilde{\to}} T^\#/P_1 \times T^\#/\bigcap_{i=2}^s P_i \cong D \times C$. The ring D is considered a T-algebra naturally and $D \cong_T T^\#/P_1$. Similarly we can see that $P_i + P_j = T^\#$ for any $i \neq j$. So consider $T^\#/P_j$ instead of D, we have a direct product decomposition:

$$\Phi : T^\# \cong T^\#/P_1 \times \cdots \times T^\#/P_s.$$

Considering $T = T \otimes_S S \otimes_S \cdots \otimes_S S \hookrightarrow T^{\sigma_1} \otimes_S \cdots \otimes_S T^{\sigma_\ell} = T^\# \to T^\#/P_i$ $(1 \leq i \leq s)$, $T^\#/P_1$ is a T-algebra $(1 \leq i \leq s)$ and Φ is a T-algebra isomorphism. Moreover each $T^\#/P_i$ is regular (and hence normal) and no non-zero element of T is a zero-divisor on $T^\#/P_i$ $(1 \leq i \leq s)$.

(2) Now we claim that

$$aD \neq D \quad (\forall a \in S \setminus S^\times) \qquad (\#).$$

Note first that for all $p \in \mathrm{Ht}_1(S)$, $pT \neq T$ because p is principal and $T^\times = k^\times$, and hence that $pT^\sigma \neq T^\sigma$ for all $\sigma \in G$. Thus $pT^\# \neq T^\#$ for all $p \in \mathrm{Ht}_1(S)$ by Remark 1.2. Since S is a polynomial ring, any $p \in \mathrm{Ht}_1(S)$ is principal.

Let $a \in S$ $(\subseteq T^\#)$ be any non-zero prime element in S. Then by the above argument, $aT^\# \neq T^\#$. When $s = 1$, then the assertion $(\#)$ holds obviously. So we may assume that $s \geq 2$.

Suppose that $a \in S$ is a prime element and that $aD = D$.

Then $aT^\# + P_1 = T^\#$ and $P_2 \cdots P_s = T^\#(P_2 \cdots P_s) = (aT^\# + P_1)(P_2 \cdots P_s) = aP_2 \cdots P_s$ because $P_1 \cdots P_s = (0)$. That is,

$$aP_2 \cdots P_s = P_2 \cdots P_s \qquad (*).$$

Throughout this proof, for a subset V of $T^\boxtimes$, $V^\times$ denotes $T^\boxtimes \cap V$,

Put $p = aS \in \mathrm{Ht}_1(S)$. Let $T_p^\# := T^\# \otimes_S S_p = T_p^{\sigma_1} \otimes_{S_p} \cdots \otimes_{S_p} T_p^{\sigma_\ell}$, (which is a semi-local ring because $S \to T^\#$ is etale). Note that the Going Up Theorem holds for $S_p \subseteq T_p$ because both S and T are integral domain and $\mathrm{ht}(p) = 1$. Since $pT^\# \neq T^\#$, we have $pT_p^\# \neq T_p^\#$.

Any prime ideal P of $T_p^\# = (S \setminus p)^{-1}T^\#$ is $(P \cap T^\#)(S \setminus p)^{-1}T^\#$, that is, there exists the canonical bijection $\mathrm{Spec}((S \setminus p)^{-1}T^\#) \cong \{Q \in \mathrm{Spec}(T^\#) | (S \setminus p) \cap Q = \emptyset\}$ corresponding $P \mapsto P \cap T^\#$.

Let M be a maximal ideal of $T_p^\#$. Then $M' = M \cap T^\#$ is a prime ideal satisfying $M' \cap (S \setminus p) = \emptyset$. So $M \cap S$ is either (0) or p.

Suppose that $M \cap S = (0)$, that is, $M' \cap S = (0)$. Then $M' \cap T = (0)$ and $\mathrm{ht}(M') = 0$ because T is algebraic over S and $S \to T^\#$ is etale. Let $T^\boxtimes = T \otimes_S \cdots \otimes_S T$ (ℓ-times) and $\lambda : T^\boxtimes \to T$ be an S-algebra homomorphism sending $c_1 \otimes \cdots \otimes c_\ell$ to $c_1 \cdots c_\ell$ with $c_i \in T$. The S-algebra $T^\boxtimes$ can be T-algebra by the canonical homomorphism $T = T \otimes_S S \otimes_S \cdots \otimes_S S \to T^\boxtimes$. Let $\Psi : T^\# \to T^\boxtimes$ be an S-isomorphism sending $c_1^{\sigma_1} \otimes \cdots \otimes c_\ell^{\sigma_\ell}$ to $c_1 \otimes \cdots \otimes c_\ell$ with $c_i \in T$ and let $\Psi_p : T_p^\# \cong T_p^\boxtimes$. Then $M_p'' = \Psi(M_p') = \Psi(M')_p$. Note that λ is an etale surjection. Put $M'' = \Psi(M')$. Then $M'' \cap T = (0)$ in $T^\boxtimes$. It is easy to see that the S-algebra homomorphisms Ψ and λ can be T-algebra homomorphisms in the natural way.

(i) If $\lambda(M'') = T$, then the restriction $\lambda| : M'' \to T$ is a split surjection as T-modules. Then $T^\boxtimes/\mathrm{Ker}(\lambda) \cong_T \lambda(M'') = T$. So $T^\boxtimes \cong_T \mathrm{Ker}(\lambda) + T$. Thus $T^\boxtimes = \mathrm{Ker}(\lambda) + mT$ for some $m \in M''$ with $\lambda(m) = 1$. Since for any $t \in T$, $\lambda(mt - t) = \lambda(m)\lambda(t) - t = t - t = 0$, we have $mT + \mathrm{Ker}(\lambda) = T + \mathrm{Ker}(\lambda)$. Note that both $\mathrm{Ker}(\lambda)$ and M'' are contained in $\{\Psi(P_1), \ldots, \Psi(P_s)\}$ since $\mathrm{ht}(\mathrm{Ker}(\lambda)) = 0 = \mathrm{ht}(M'')$. So $M'' = \Psi(P_i)$ and $\mathrm{Ker}(\lambda) = \Psi(P_j)$. Since $\lambda(M'') = T$, we may assume that $\Psi(P_2) = \mathrm{Ker}(\lambda)$, otherwise $D \cong_T T^\boxtimes/\mathrm{Ker}(\lambda) \cong_T T$ and $D^\times = T^\times = k^\times$, a contradiction. Let $\Psi' : T^\boxtimes \cong_T T^\boxtimes/M'' \times T^\boxtimes/\mathrm{Ker}(\lambda) \times T^\boxtimes/\Psi(P_3) \times \cdots \times T^\boxtimes/\Psi(P_s)$ be the isomorphism induced from Ψ. Then $\lambda(M'') = T = \lambda\Psi'^{-1}(T^\boxtimes/\mathrm{Ker}(\lambda)) = \lambda(T^\boxtimes) = \lambda(\Psi'^{-1}((T^\boxtimes/M'') \times (T^\boxtimes/\mathrm{Ker}(\lambda)) \times (T^\boxtimes/\Psi(P_3)) \times \cdots \times (T^\boxtimes/\Psi(P_s)))$. Hence we have $\Psi'^{-1}((T^\boxtimes/\Psi(M'') \times 0 \times (T^\boxtimes/\Psi(P_3) \times \cdots \times T^\boxtimes/\Psi(P_s)) \subseteq \mathrm{Ker}(\lambda)$, but since $\lambda(M'') = T$ and $\mathrm{Ker}(\lambda)$ is a prime ideal of $T^\boxtimes$, $\Psi^{-1}(T^\boxtimes/M'')$ must be $\mathrm{Ker}(\lambda)$. Thus $T^\boxtimes/\Psi(P_3) \times \cdots \times T^\boxtimes/\Psi(P_s) = 0$, which means that $s = 2$ in this

case. So $M''\cap\Psi(P_2) = (0)$. Moreover $T^{\boxtimes} = \mathrm{Ker}(\lambda)+mT = \mathrm{Ker}(\lambda)+M''$, it follows that $T^{\boxtimes} \cong_T T^{\boxtimes}/M'' \times T^{\boxtimes}/\mathrm{Ker}(\lambda) = T^{\boxtimes}/M'' \times T^{\boxtimes}/\Phi(P_2)$. Thus $D \cong_T T^{\boxtimes}/M''$. Thus

$$\Psi' : T^{\boxtimes} \cong_T T^{\boxtimes}/M'' \times T^{\boxtimes}/\mathrm{Ker}(\lambda) \qquad (**)$$

whence $M'' = \Psi(P_1)$ and $\mathrm{Ker}(\lambda) = \Psi(P_2)$. Here in this case, $s = 2$ in (1). We have $(T^{\boxtimes})^\times \subseteq mT^\times+\mathrm{Ker}(\lambda) = mk^\times+\mathrm{Ker}(\lambda) = k^\times+\mathrm{Ker}(\lambda)$. Note that $(k^\times+\mathrm{Ker}(\lambda))^\times$ is a group by the multiplication in $T^{\boxtimes}$. Thus

$$(T^{\boxtimes})^\times \subseteq (k^\times + \mathrm{Ker}(\lambda))^\times \qquad (***)$$

From $(**)$, we have

$$k^\times \times k^\times \subseteq (T^{\boxtimes})^\times \cong (T^{\boxtimes}/M'')^\times \times (T^{\boxtimes}/\mathrm{Ker}(\lambda))^\times \qquad (****)$$

It is easy to see that $(T^{\boxtimes}/\mathrm{Ker}(\lambda))^\times \cong T^\times = k^\times$, and so that by $(****)$ and $(***)$ we have $(T^{\boxtimes}/M'')^\times \subseteq (k^\times+\mathrm{Ker}(\lambda))^\times/k^\times \subseteq (1+\mathrm{Ker}(\lambda))^\times$, that is $k^\times \subseteq (1+\mathrm{Ker}(\lambda))^\times$, which is impossible because if we take any $c \in k^\times$ then $1-c \in \mathrm{Ker}(\lambda)\cap k$ implies $c = 1$. So this case does not occur.

(ii) If $\lambda(M'')\cap S = p$, then it is easy to see that $M'\cap S = p$, a contradiction.

(iii) Let $\lambda(M'')\cap S = (0)$. In this case, $\lambda(M'') = (0)$ because $S \hookrightarrow T$ is algebraic, and hence $M'' \subseteq \mathrm{Ker}(\lambda)$. We have an S_p-isomorphism $T^{\boxtimes}_p/\mathrm{Ker}(\lambda)_p \cong T_p$. Since $pT_p \neq T_p$, there exists a prime ideal N'' of $T^{\boxtimes}$ such that $N'' \supset M''$, $\mathrm{ht}(N'') = 1$ and $N''\cap S = p$ because λ is etale. So $N' := \Psi^{-1}(\lambda^{-1}(N''))$ satisfies $N'_p \supsetneq M'_p = M$ and $N'_p \cap S = p$ because λ is etale, which contradicts the maximality of M.

Therefore $M'\cap S = M\cap S = p$.

So we conclude that the Jacobson radical $J(T^{\#}_p)$ of $T^{\#}_p$ is $\sqrt{pT^{\#}_p}$ and contains the prime element a.

From $(*)$, we have $aP_{2p}\cdots P_{sp} = P_{2p}\cdots P_{sp}$, which is a finitely generated $T^{\#}_p$-module. Thus there exists $\beta \in T^{\#}_p$ such that $(1-a\beta)P_{2p}\cdots P_{sp} = 0$. Since a is contained in the Jacobson radical $J(T^{\#}_p)$ of the semi-local ring $T^{\#}_p$ as mentioned above, we have $P_{2p}\cdots P_{sp} = 0$. Since any element of $S\setminus p$ is not a zero-divisor on $T^{\#}$, we have $P_2\cdots P_s \subseteq P_{2p}\cdots P_{sp} = (0)$. So $P_2\cap\cdots\cap P_s = P_2\cdots P_s = (0)$. But $(0) = P_1\cap\cdots\cap P_s$ is an irredundant primary decomposition as mentioned above, which is a **contradiction**. Hence $(\#)$ has been proved.

(3) Let C be the integral closure of S in L. Then $C \subseteq D$ because D is regular (hence normal) and C is an k-affine domain (Lemma D). For any $\sigma \in G = G(L/K(S))$, $C^\sigma \subseteq D$ because C^σ is integral over S and D is normal with $K(C) = L$. Hence $C^\sigma = C$ for any $\sigma \in G$. Note that both D and C have the quotient field L. Zarisiki's Main Theorem(Lemma C) yields the decomposition:

$$\mathrm{Spec}(D) \xrightarrow{\ \ i\ \ } \mathrm{Spec}(C) \xrightarrow{\ \ \pi\ \ } \mathrm{Spec}(S),$$

where i is an open immersion and π is integral(finite). We identify $\mathrm{Spec}(D) \hookrightarrow \mathrm{Spec}(C)$ as open subset and $D_P = C_{P \cap C}$ ($P \in \mathrm{Spec}(D)$). Let $Q \in \mathrm{Ht}_1(C)$ with $Q \cap S = p = aS$. Then Q is a prime divisor of aC. Since $aD \neq D$ by (#) in (2), there exists $P \in \mathrm{Ht}_1(D)$ such that $P \cap S = p$. Hence there exists $\sigma \in G$ such that $Q = (P \cap C)^\sigma$ because any minimal divisor of aC is $(P \cap C)^{\sigma'}$ for some $\sigma' \in G$ (**[9,(5.E)]**), noting that C is a Galois extension of S. Since $D_P = C_{P \cap C}$ is unramified over $S_{P \cap S} = S_p$, $C_Q = C_{(P \cap C)^\sigma} \cong C_{(P \cap C)}$ is unramified over S_p. Hence C is unramified over S by Lemma I. By Corollary A.1, C is finite etale over S. So Proposition 1.1 implies that $C = S$. In particular, $L = K(D) = K(C) = K(S)$ and hence $K(T) = K(S)$. Since $S \hookrightarrow T$ is birational etale, $\mathrm{Spec}(T) \hookrightarrow \mathrm{Spec}(S)$ is an open immersion by Lemma C. Let J be an ideal of S such that $V(J) = \mathrm{Spec}(S) \setminus \mathrm{Spec}(T)$. Suppose that $J \neq S$. Then $V(J)$ is pure of codimension one by Lemma C. Hence J is a principal ideal aS because S is a UFD. Since $JT = aT = T$, a is a unit in T. But $T^\times = k^\times$ implies that $a \in k^\times$ and hence that $J = S$, a contradiction. Hence $V(J) = \emptyset$, that is, $T = S$. **Q.E.D.**

3. The Jacobian Conjecture

The Jacobian conjecture has been settled affirmatively in several cases. For example,
Case(1) $k(X_1, \ldots, X_n)$ is a Galois extension of $k(f_1, \ldots, f_n)$ (cf. **[4]**,**[6]** and **[15]**);
Case(2) $\deg f_i \leq 2$ for all i (cf. **[13]** and **[14]**);
Case(3) $k[X_1, \ldots, X_n]$ is integral over $k[f_1, \ldots, f_n]$. (cf. **[4]**).

A general reference for the Jacobian Conjecture is **[4]**.

Remark 3.1. (1) In order to prove Theorem 3.2, we have only to show that the inclusion $k[f_1, \ldots, f_n] \longrightarrow k[X_1, \ldots, X_n]$ is surjective. For this it suffices that $k'[f_1, \ldots, f_n] \longrightarrow k'[X_1, \ldots, X_n]$ is surjective, where k' denotes an algebraic closure of k. Indeed, once we proved $k'[f_1, \ldots, f_n] = k'[X_1, \ldots, X_n]$, we can write for each $i = 1, \ldots, n$:

$$X_i = F_i(f_1, \ldots, f_n),$$

where $F_i(Y_1, \ldots, Y_n) \in k'[Y_1, \ldots, Y_n]$, a polynomial ring in Y_i. Let L be an intermediate field between k and k' which contains all the coefficients of F_i and is a finite Galois extension of k. Let $G = G(L/k)$ be its Galois group and put $m = \#G$. Then G acts on a polynomial ring $L[X_1, \ldots, X_n]$ such that $X_i^g = X_i$ for all i and all $g \in G$ that is, G acts on coefficients of an element in $L[X_1, \ldots, X_n]$. Hence

$$mX_i = \sum_{g \in G} X_i^g = \sum_{g \in G} F_i^g(f_1^g, \ldots, f_n^g) = \sum_{g \in G} F_i^g(f_1, \ldots, f_n).$$

Since $\sum_{g \in G} F_i^g(Y_1, \ldots, Y_n) \in k[Y_1, \ldots, Y_n]$, it follows that $\sum_{g \in G} F_i^g(f_1, \ldots, f_n) \in k[f_1, \ldots, f_n]$. Therefore $X_i \in k[f_1, \ldots, f_n]$ because L has a characteristic zero. So we may assume that k is algebraically closed.

(2) Let k be a field, let $k[X_1, \ldots, X_n]$ denote a polynomial ring and let $f_1, \ldots, f_n \in k[X_1, \ldots, X_n]$. If the Jacobian $\det\left(\dfrac{\partial f_i}{\partial X_i}\right) \in k^\times (= k \setminus (0))$, then the $k[X_1, \ldots, X_n]$ is unramified over the subring $k[f_1, \ldots, f_n]$. Consequently $f_1, \ldots, f_n$ is algebraically independent over k.

In fact, put $T = k[X_1, \ldots, X_n]$ and $S = k[f_1, \ldots, f_n] (\subseteq T)$. We have an exact sequence by **[9, (26.H)]** :

$$\Omega_{S/k} \otimes_S T \xrightarrow{v} \Omega_{T/k} \longrightarrow \Omega_{T/S} \longrightarrow 0,$$

where

$$v(df_i \otimes 1) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j} dX_j \qquad (1 \leq i \leq n).$$

So det $\left(\dfrac{\partial f_i}{\partial X_j}\right) \in k^\times$ implies that v is an isomorphism. Thus $\Omega_{T/S} = 0$ and hence T is unramified over S by **[2, VI,(3.3)]** or **[9]**. Moreover $K(T)$ is algebraic over $K(S)$, which means that $f_1, \ldots, f_n$ are algebraically independent over k.

As a result of Theorem 2.1, we have the following.

Theorem 3.2 (The Jacobian Conjecture)**.** *Let k be a field of characteristic zero, let $k[X_1, \ldots, X_n]$ be a polynomial ring over k, and let $f_1, \ldots, f_n$ be elements in $k[X_1, \ldots, X_n]$. Then the Jacobian matrix $(\partial f_i/\partial X_j)$ is invertible if and only if $k[X_1, \ldots, X_n] = k[f_1, \ldots, f_n]$.*

4. Generalization of The Jacobian Conjecture

The Jacobian Conjecture (Theorem 3.2) can be generalized as follows.

Theorem 4.1. *Let A be an integral domain whose quotient field $K(A)$ is of characteristic zero. Let $f_1, \ldots, f_n$ be elements of a polynomial ring $A[X_1, \ldots, X_n]$ such that the Jacobian determinant det$(\partial f_i/\partial X_j)$ is a unit in A. Then*

$$A[X_1, \ldots, X_n] = A[f_1, \ldots, f_n].$$

Proof. It suffices to prove $X_1, \ldots, X_n \in A[f_1, \ldots, f_n]$. We have $K(A)[X_1, \ldots, X_n] = K(A)[f_1, \ldots, f_n]$ by Theorem 3.2. Hence

$$X_1 = \sum c_{i_1 \cdots i_n} f_1^{i_1} \cdots f_n^{i_n}$$

with $c_{i_1 \cdots i_n} \in K(A)$. If we set $f_i = a_{i1}X_1 + \ldots + a_{in}X_n +$ (higher degree terms), $a_{ij} \in A$, then the assumption implies that the determinant of a matrix (a_{ij}) is a unit in A. Let

$$Y_i = a_{i1}X_1 + \ldots + a_{in}X_n \quad (1 \le i \le n).$$

Then $A[X_1, \ldots, X_n] = A[Y_1, \ldots, Y_n]$ and $f_i = Y_i +$ (higher degree terms). So to prove the assertion, we can assume that without loss of generality the linear parts of $f_1, \ldots, f_n$ are $X_1, \ldots, X_n$, respectively. Now we introduce a linear order in the set $\{(i_1, \ldots, i_n) \mid i_k \in \mathbf{Z}\}$ of lattice points in $\mathbf{R}^n$ (where $\mathbf{R}$ denotes the field of real

numbers) in the way : $(i_1, \ldots, i_n) > (j_1, \ldots, j_n)$ if (1) $i_1 + \ldots + i_n > j_1 + \ldots + j_n$ or (2) $i_1 + \ldots + i_k > j_1 + \ldots + j_k$ and $i_1 + \ldots + i_{k+1} = j_1 + \ldots + j_{k+1}, \ldots\ldots, i_1 + \ldots + i_n = j_1 + \ldots + j_n$. We shall show that every $c_{i_1 \ldots i_n}$ is in A by induction on the linear order just defined. Assume that every $c_{j_1 \ldots j_n}$ with $(j_1, \ldots, j_n) < (i_1, \ldots, i_n)$ is in A. Then the coefficients of the polynomial

$$\sum c_{j_1 \cdots j_n} f_1^{j_1} \cdots f_n^{j_n}$$

are in A, where the summation ranges over $(j_1, \ldots, j_n) \geq (i_1, \ldots, i_n)$. In this polynomial, the term $X_1^{i_1} \cdots X_n^{i_n}$ appears once with the coefficient $c_{i_1 \ldots i_n}$. Hence $c_{i_1 \ldots i_n}$ must be an element of A. So X_1 is in $A[f_1, \ldots, f_n]$. Similarly $X_2, \ldots, X_n$ are in $A[f_1, \ldots, f_n]$ and the assertion is proved completely. $\square$

Corollary 4.2. (Keller's Problem) *Let $f_1, \ldots, f_n$ be elements of a polynomial ring $\mathbf{Z}[X_1, \ldots, X_n]$ over $\mathbf{Z}$, the ring of integers. If the Jacobian determinant $det(\partial f_i / \partial X_j)$ is equal to either ± 1, then $\mathbf{Z}[X_1, \ldots, X_n] = \mathbf{Z}[f_1, \ldots, f_n]$.*

References

[1] S.Abhyankar, Expansion technique in algebraic geometry, Tata Institute of Fundamental Research, Springer-Verlag (1977).

[2] A.Altman and S.Kleiman, Introduction to Grothendieck duality theory, Lecture Notes in Math. 146, Springer-Verlag (1970).

[3] M.F.Atiyah and L.G.MacDonald, Introduction to Commutative Algebra, Addison-Wesley, London (1969).

[4] H.Bass, E.H.Connel and D.Wright, The Jacobian conjecture, Bull. A.M.S., 7(2) (1982),287-330.

[5] N.Bourbaki, Commutative Algebra, translated from the French, Herman, Paris; Addison-Wesley, Reading, Mass. (1972).

[6] L.A.Campbell, A condition for a polynomial map to be invertible, Math. Ann., 205 (1973),243-248.

[7] E.Kung, Introduction to Commutative Algebra and Algebraic Geometry, Birdhouses (1985).

[8] O.H.Keller, Gamze Cremona-Transformationen, Monatsheft fur Math. und Phys., Vol. 47 (1939),299-306.

[9] H.Matsumura, Commutative Algebra, Benjamin, New York (1970).

[10] M.Nagata, Lectures on The Fourteenth Problem of Hilbert, Tata Institute of Fundamental Research, Bombay (1965).

[11] M.Nagata, Local Rings, Interscience (1962).

[12] M.Raynaud, Anneaux Locaux Henséliens, Lecture Notes in Math. 169, Springer-Verlag (1970).

[13] S.Oda and K.Yoshida, A short proof of the Jacobian conjecture in the case of degree ≤ 2, C.R. Math. Rep., Acad. Sci. Canada, Vol.V (1983),159-162.
[14] S.Wang, On the Jacobian conjecture, J. Algebra 65 (1980), 453-494.
[15] D.Wright, On the Jacobian conjecture, Illinois J. Math., 25 (1981), 423-440.
[16] O.Zariski and P.Samuel, Commutative Algebra, Vol. I GTM 28, Springer-Verlag (1979).
[17] M.Nagata, M.Miyanishi and M.Maruyama : Abstract Algebraic Geometry, (in Japanese), Kyoritsu-shuppan (1972).

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